

Binary Locally Repairable Codes —Sequential Repair for Multiple Erasures

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Abstract—Locally repairable codes (LRC) for distributed storage allow two approaches to locally repair multiple failed nodes: 1) parallel approach, by which each newcomer access a set of r live nodes (r is the repair locality) to download data and recover the lost packet; and 2) sequential approach, by which the newcomers are properly ordered and each newcomer access a set of r other nodes, which can be either a live node or a newcomer ordered before it. An $[n, k]$ linear code with locality r and allows local repair for up to t failed nodes by sequential approach is called an (n, k, r, t) -exact locally repairable code (ELRC).

In this paper, we present a family of binary codes which is equivalent to the direct product of m copies of the $[r+1, r]$ single-parity-check code. We prove that such codes are (n, k, r, t) -ELRC with $n = (r+1)^m$, $k = r^m$ and $t = 2^m - 1$, which implies that they permit local repair for up to $2^m - 1$ erasures by sequential approach. Our result shows that the sequential approach has much bigger advantage than parallel approach.

I. INTRODUCTION

In a distributed storage system (DSS), data is stored through a large, distributed network of storage nodes. To maintain the data reliability in the presence of node failures, the system should have the ability of *node repair*. That is, when some of the storage nodes fail, each failed node is replaced by a *newcomer* where the lost packet is recovered and stored again.

Various coding techniques are employed by modern DSS to improve system performance, among which locally repairable codes (LRC) aim to minimize the repair locality, i.e. the number of disk accesses required for single node repair [1]–[4].

The i th coordinate of an $[n, k]$ linear code \mathcal{C} (also called the i th code symbol of \mathcal{C}) is said to have *locality* r , if its value is computable from the values of a set of at most r other coordinates of \mathcal{C} (called a repair set of i). Codes with all code symbols having locality r ($r < k$) are called locally repairable codes. In a DSS with an LRC \mathcal{C} as the storage code, the data packet stored in each storage node is a code symbol of \mathcal{C} and any single failed node can be “locally and exactly repaired” in the sense that the newcomer can recover the lost data by accessing at most r other nodes, where r is the locality of \mathcal{C} .

To handle the problem of local repair for multiple failed nodes, some special subclasses of LRCs are investigated, such as: a) Codes with all-symbol locality $(r, t+1)$, also called $(r, t+1)_a$ codes, in which each code symbol is contained in a local code of length at most $r+t$ and minimum distance at least $t+1$ [6]; b) Codes with all-symbol locality r and availability t , in which each code symbol has t pairwise disjoint repair sets

with locality r [7], [8]; c) Codes with (r, t) -locality, in which each subset of t code symbols can be cooperatively repaired from at most r other code symbols [9] (For convenience, in the following, we will call such codes as (r, t) -CLRC.); d) Codes with overall local repair tolerance t , in which for any $E \subseteq [n]$ of size t and any $i \in E$, the i th code symbol has a repair set contained in $[n] \setminus E$ and with locality r [5]. Clearly, these four subclasses of LRC permit local repair for up to t failed nodes by *parallel approach* — each newcomer can access r live nodes to recover the corresponding lost packet. We also call t as the erasure tolerance of such codes.

For $(r, \delta)_a$ codes and (r, t) -CLRC, the code rate satisfies (e.g., see [13] and [9]):

$$\frac{k}{n} \leq \frac{r}{r+t}. \quad (1)$$

For codes with locality r and availability t , it was proved in [10] that the code rate satisfies:

$$\frac{k}{n} \leq \frac{1}{\prod_{j=1}^t (1 + \frac{1}{j^r})}. \quad (2)$$

However, for $t \geq 2$, it is not known whether the code rate bound (2) is achievable. Recent work by Wang et al. [11] shows that for any positive integers r and t , there exist codes with locality r and availability t over the binary field with code rate $\frac{r}{r+t}$. Unfortunately, such codes do not achieve the bound (2) for $t \geq 2$. The problem of constructing codes with locality r and availability $t \geq 2$ that achieve the optimal code rate is still an open problem.

A more general way to locally repair t ($t \geq 2$) failed nodes is the *sequential approach*, by which the t newcomers can be properly ordered in a sequence and, to recover the lost packet, each newcomer can access r other nodes, each of which can be a live node or a newcomer ordered before it [14], [15]. In [15], an $[n, k]$ linear code that has locality r and permit local repair for up to t failed nodes by sequential approach is called an (n, k, r, t) -exact locally repairable code (ELRC). Clearly, the four subclasses of LRC, i.e., $(r, \delta)_a$ codes, (r, t) -CLRC, codes with locality r and availability t , and codes with overall local repair tolerance t , are all (n, k, r, t) -ELRC. Potentially, the sequential approach allows us to design codes with improved parameter properties than the parallel approach.

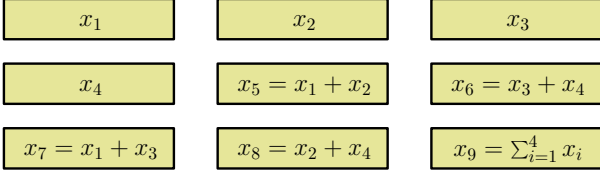


Fig. 1. A $(9, 4, 2, 3)$ -ELRC.

Example 1: As an example of sequential approach, consider the code illustrated in Fig. 1, where x_1, \dots, x_4 are information symbols and x_5, \dots, x_9 are parity symbols. We can check that $x_1 = x_2 + x_5 = x_3 + x_7$. So $\{x_2, x_5\}$ and $\{x_3, x_7\}$ are two disjoint repair sets of x_1 . Similarly, we can find two disjoint repair sets for each of x_2, \dots, x_9 . The repair set of each code symbol is illustrated in Fig. 2. So this code has locality 2 and availability 2. Hence, it permits local repair for up to 2 erasures by the parallel approach.

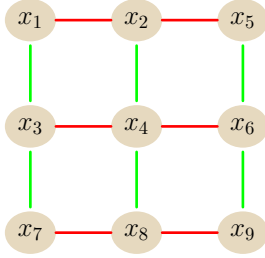


Fig. 2. Repair relation of code symbols of the code in Fig. 1: Each line (red line or green line) contains 3 symbols and any symbol on a line can be computed from the other symbols on the same line. Note that each symbol belongs to two lines — a red line and a green line, hence has two repair sets.

However, we can check that this code is a $(9, 4, 2, 3)$ -ELRC — it permits local repair for up to 3 failed nodes by sequential approach. For example, if x_1, x_5, x_7 are lost, then we can repair them by the following sequence of equations: $x_5 = x_6 + x_9$, $x_7 = x_8 + x_9$ and $x_1 = x_2 + x_5$. During the repair process, x_5 is repaired before x_1 . Once x_5 is repaired, it can be used to repair x_1 . Hence, the repair process is feasible. Note that x_1, x_5, x_7 can't be repaired by parallel approach because both the two repair sets of x_1 contain a lost symbol.

Most existing works about LRC focus on parallel repair approach [5]–[13]. In the field of (n, k, r, t) -ELRC (i.e., LRC with sequential repair approach), only for $t \in \{2, 3\}$ is investigated [14], [15].

For $(n, k, r, t = 2)$ -ELRC, the code rate satisfies [14]:

$$\frac{k}{n} \leq \frac{r}{r+2}. \quad (3)$$

An upper bound for the minimum distance of such codes and a construction of codes achieving the minimum distance bound were also given in [14].

For $(n, k, r, t = 3)$ -ELRC, it was proved in [15] that the code length n satisfies:

$$n \geq k + \left\lceil \frac{2k + \lceil \frac{k}{r} \rceil}{r} \right\rceil \quad (4)$$

and there exist codes with code length meet this bound. However, for $t \geq 3$, no result is known about the minimum distance bound, and for $t \geq 4$, no result is known about the code rate bound. Construction of $(n, k, r, t \geq 4)$ -ELRC is not seen in literature either.

In this paper, we prove that for any given positive integers r ($r \geq 2$) and m , the direct product of m copies of the $[r+1, r]$ single-parity-check code is an $(n, k, r, t = 2^m - 1)$ -ELRC. So such code permits local repair for up to $t = 2^m - 1$ erasures by sequential approach. The code rate of such codes is shown to be much larger than $(r, \delta)_a$ codes and (r, t) -CLRC. Moreover, it was pointed out in [10] that such code has locality r and availability m , which implies that it permits local repair for up to only m failed nodes by parallel approach. Hence, our result shows that sequential approach has much bigger advantage than parallel approach for such codes.

The rest of this paper is organized as follows. In Section II, we state the formal definition of (n, k, r, t) -ELRC. In section III, we give a method to construct codes that are equivalent to the direct product codes and present our main theorem. We prove the main theorem in Section IV and conclude the paper in Section V.

II. PRELIMINARY

For any set A , we use $|A|$ to denote the size (i.e., the number of elements) of A . A set B is called an r -subset of A if $B \subseteq A$ and $|B| = r$. For any positive integer n , we denote

$$[n] := \{1, 2, \dots, n\}.$$

An $[n, k]$ linear code over the finite field \mathbb{F} is a k -dimensional subspace of the vector space \mathbb{F}^n , where n, k are positive integers and $k \leq n$.

In this section, we present the formal definition of (n, k, r, t) -exact locally repairable code (ELRC). More details can be found in [15].

Let \mathcal{C} be an $[n, k]$ linear code over the field \mathbb{F} . If there is no confusion in the context, we will omit the base field \mathbb{F} and only say that \mathcal{C} is an $[n, k]$ linear code. A k -subset S of $[n]$ is called an *information set* of \mathcal{C} if for all codeword $x = (x_1, x_2, \dots, x_n) \in \mathcal{C}$ and all $i \in [n]$, $x_i = \sum_{j \in S} a_{i,j} x_j$, where all $a_{i,j} \in \mathbb{F}$ and are independent of x . The code symbols in $\{x_j, j \in S\}$ are called *information symbol* of \mathcal{C} . In contrast, code symbols in $\{x_i, i \in [n] \setminus S\}$ are called *parity symbol* of \mathcal{C} . An $[n, k]$ linear code has at least one information set.

Definition 2: Let $i \in [n]$ and $R \subseteq [n] \setminus \{i\}$. The subset R is called an (r, \mathcal{C}) -*repair set* of i if $|R| \leq r$ and $x_i = \sum_{j \in R} a_{i,j} x_j$ for all $x = (x_1, x_2, \dots, x_n) \in \mathcal{C}$, where all $a_{i,j} \in \mathbb{F}$ and are independent of x .

Definition 3: Let E be a t -subset of $[n]$ and $\overline{E} = [n] \setminus E$. The code \mathcal{C} is said to be (E, r) -repairable if there exists an index of E , say $E = \{i_1, \dots, i_t\}$, and a collection of subsets

$$\{R_\ell \subseteq \overline{E} \cup \{i_1, \dots, i_{\ell-1}\}; |R_\ell| \leq r, \ell \in [t]\}$$

such that for each $\ell \in [t]$, R_ℓ is an (r, \mathcal{C}) -repair set of i_ℓ .

Definition 4: An (n, k, r, t) -exact locally repairable code (ELRC) is an $[n, k]$ linear code \mathcal{C} such that for each $E \subseteq [n]$ of size $|E| \leq t$, \mathcal{C} is (E, r) -repairable.

By Definition 3 and 4, if a DSS uses an (n, k, r, t) -ELRC as the storage code, then any t' ($t' \leq t$) failed nodes can be locally repaired by sequential approach.

The following lemma gives a seemingly simpler characterization for (n, k, r, t) -ELRC.

Lemma 5 ([15], Lemma 6): An $[n, k]$ linear code \mathcal{C} is an (n, k, r, t) -ELRC if and only if for any $E \subseteq [n]$ of size $0 < |E| \leq t$, there exists an $i \in E$ such that i has an (r, \mathcal{C}) -repair set contained in $[n] \setminus E$.

In the following, if R is an (r, \mathcal{C}) -repair set of i , we will omit the prefix (r, \mathcal{C}) and only say that R is a repair set of i .

III. CODE CONSTRUCTION

Let r, m be two positive integers such that $r \geq 2$. Let $n = (r+1)^m$ and $k = r^m$. We will construct a binary $[n, k]$ linear code that is equivalent to the direct product of m copies of the $[r+1, r]$ single-parity-check code. Moreover, we will show that such code is an (n, k, r, t) -ELRC, where $t = 2^m - 1$.

In the following, we will denote

$$\mathbb{Z}_r = \{0, 1, \dots, r-1\}$$

and

$$\mathbb{Z}_r^m = \{(\lambda_1, \dots, \lambda_m); \lambda_1, \dots, \lambda_m \in \mathbb{Z}_r\}.$$

That is, \mathbb{Z}_r^m is the Cartesian product of m copies of \mathbb{Z}_r . Similarly, we denote

$$\mathbb{Z}_{r+1} = \{0, 1, \dots, r\}$$

and

$$\mathbb{Z}_{r+1}^m = \{(\lambda_1, \dots, \lambda_m); \lambda_1, \dots, \lambda_m \in \mathbb{Z}_{r+1}\}.$$

Then $\mathbb{Z}_r \subseteq \mathbb{Z}_{r+1}$ and $\mathbb{Z}_r^m \subseteq \mathbb{Z}_{r+1}^m$. To describe the code construction method, we need the following two notations:

For each $\alpha = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{r+1}^m \setminus \mathbb{Z}_r^m$, denote

$$T(\alpha) = \{j \in [m]; \lambda_j \in \mathbb{Z}_r\} \quad (5)$$

and

$$\mathcal{L}(\alpha) = \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}_r^m; \mu_j = \lambda_j, \forall j \in T(\alpha)\}. \quad (6)$$

For example, let $r = 2, m = 6$. For $\alpha = (0, 1, 2, 0, 2, 2) \in \mathbb{Z}_3^6$, we have $T(\alpha) = \{1, 2, 4\}$ and

$$\mathcal{L}(\alpha) = \{(0, 1, \lambda_3, 0, \lambda_5, \lambda_6); \lambda_3, \lambda_5, \lambda_6 \in \mathbb{Z}_2\}.$$

Clearly, for each $\alpha = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{r+1}^m \setminus \mathbb{Z}_r^m$, $T(\alpha)$ is a proper subset of $[m]$ and $\mathcal{L}(\alpha)$ is a non-empty subset of \mathbb{Z}_r^m . Moreover, if $\alpha = (r, \dots, r)$, then $T(\alpha) = T(r, \dots, r) = \emptyset$ and $\mathcal{L}(\alpha) = \mathcal{L}(r, \dots, r) = \mathbb{Z}_r^m$.

Let $n = (r+1)^m$ and $k = r^m$. Let $H = (h_{\alpha, \beta})$ be an $(n-k) \times n$ binary matrix whose rows are indexed by $\mathbb{Z}_{r+1}^m \setminus \mathbb{Z}_r^m$ and columns are indexed by \mathbb{Z}_{r+1}^m such that

$$h_{\alpha, \beta} = \begin{cases} 1, & \text{if } \beta \in \mathcal{L}(\alpha) \cup \{\alpha\}; \\ 0, & \text{Otherwise.} \end{cases} \quad (7)$$

Clearly, the submatrix H_1 formed by the columns of H that are indexed by $\mathbb{Z}_{r+1}^m \setminus \mathbb{Z}_r^m$ is a permutation matrix of order $n - k$. So $\text{rank}(H) = n - k$.

Let \mathcal{C} be the binary code with a parity check matrix H . Then \mathcal{C} is an $[n, k]$ linear code. Clearly, \mathcal{C} is just the $[r+1, r]$ single-parity-check code for $m = 1$ and the square code constructed in [7] for $m = 2$. In general, it is not difficult to prove that the code \mathcal{C} is equivalent to the direct product of m copies of the $[r+1, r]$ single-parity-check code. Moreover, we have the following theorem.

Theorem 6: The code \mathcal{C} which has a parity check matrix H is an (n, k, r, t) -ELRC, where $t = 2^m - 1$.

It is easy to see that the code rate of (n, k, r, t) -ELRC obtained by the above construction is much larger than the bound (1). Comparison of code length of (n, k, r, t) -ELRC with $(r, t+1)_a$ codes and (r, t) -CLRC for $r = 2$ and $m \in \{2, 3, 4, 5\}$ is given in Table 1, from which we can see that the code rate of (n, k, r, t) -ELRC is much larger than $(r, t+1)_a$ codes and (r, t) -CLRC for the same r and t .

Moreover, it was pointed out in [10] that the direct product of m copies of the $[r+1, r]$ single-parity-check code has locality r and availability m , which implies that \mathcal{C} permits local repair for up to m erasures by parallel approach. Note that Theorem 6 shows that \mathcal{C} permits locally repair for up to $2^m - 1$ erasures by the sequential approach, which is much larger than m for $m \geq 2$. Hence, our result shows that sequential approach has much bigger advantage than parallel approach for LRC. Table 2 is the comparison of erasure tolerance of the constructed code for sequential approach and parallel approach, where we assume $r = 2$.

m	t	k	Code length of (r, t) -ELRC	Code length of $(r, t+1)_a$ codes	Code length of (r, t) -CLRC
2	3	4	9	≥ 10	≥ 10
3	7	8	27	≥ 36	≥ 36
4	15	16	81	≥ 136	≥ 136
5	31	32	243	≥ 528	≥ 528

Table 1. Comparison of code length of three subclasses of LRCs for $r = 2$.

m	k	n	Erasure tolerance by sequential repair approach	Erasure tolerance by parallel repair approach
2	4	9	3	2
3	8	27	7	3
4	16	81	15	4
5	32	243	31	5

Table 2. Comparison of erasure tolerance of the constructed code with $r = 2$: sequential approach and parallel approach.

The proof of Theorem 6 will be given in the next section. We now give an example of the above construction.

Example 7: Let $r = 2$ and $m = 3$. Then $k = 8$ and $n = 27$. We can construct a matrix H and a binary $[27, 8]$ linear code \mathcal{C} by the above method. Similar to Fig. 2, we can illustrate the repair set of each code symbol of \mathcal{C} by Fig. 3. More details can

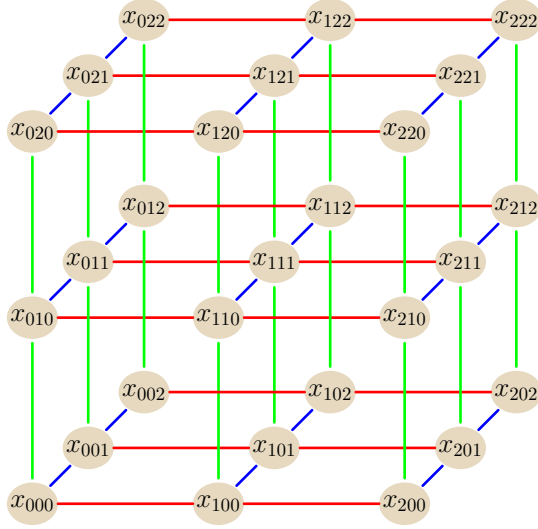


Fig 3. Repair relation of code symbols of the code in Example 7. We use \mathbb{Z}_3^3 to index the coordinates and, to simplify notation, use $x_{\lambda_1, \lambda_2, \lambda_3}$ to denote the code symbol $x_{(\lambda_1, \lambda_2, \lambda_3)}$ for each $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}_3^3$.

be seen in Lemma 8. We will show that \mathcal{C} is an $(n, k, r, t = 7)$ -ELRC. That is, any $t' \leq 7$ code symbols of \mathcal{C} can be locally repaired by other code symbols of \mathcal{C} . For example, suppose $E = \{(020), (120), (010), (110), (021), (121), (011)\}$. Then the code symbols in E can be locally repaired by the following sequence of equation: $x_{011} = x_{111} + x_{211}$, $x_{121} = x_{111} + x_{101}$, $x_{021} = x_{121} + x_{221}$, $x_{020} = x_{021} + x_{022}$, $x_{120} = x_{121} + x_{122}$, $x_{010} = x_{011} + x_{012}$ and $x_{110} = x_{111} + x_{112}$. In general, this claim can be checked as follows.

We partition the index set $\mathbb{Z}_{r+1}^3 = \mathbb{Z}_3^3$ into three subsets

$$I_j = \{(\lambda_1, \lambda_2, \lambda_3); \lambda_1, \lambda_2 \in \mathbb{Z}_{r+1} \text{ and } \lambda_3 = j\}, j = 0, 1, 2.$$

For example, $I_0 = \{(000), (010), (020), (100), (110), (120), (200), (210), (220)\}$. For each $j \in \{0, 1, 2\}$, from Fig. 3, the repair relation of code symbols in I_j is the same as code symbols in Fig. 2. So any $t' \leq 3$ code symbols in I_j can be locally repaired by other code symbols in I_j . Now, suppose $E \subseteq \mathbb{Z}_3^3$ of size $|E| \leq 7$. Then there exist at most one $j \in \{0, 1, 2\}$ such that $|E \cap I_j| > 3$. For those j such that $|E \cap I_j| \leq 3$, code symbols in $E \cap I_j$ can be locally repaired by other code symbols in $E \cap I_j$. Finally, if there exist a j_0 such that $|E \cap I_{j_0}| > 3$. Then each code symbol in $E \cap I_{j_0}$ can be locally repaired by code symbols in $I_{j_1} \cup I_{j_2}$, where $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{j_0\}$. Hence, all code symbols in E can be locally repaired by sequential approach.

IV. PROOF OF THEOREM 6

In this section, we prove Theorem 6. The basic idea of the proof is the same as in Example 7.

Before proving Theorem 6, we first need to prove a lemma. For each $\alpha = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{r+1}^m$ and each $i \in [m]$, denote

$$L_\alpha^{(i)} = \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}_{r+1}^m; \mu_j = \lambda_j, \forall j \in [m] \setminus \{i\}\}. \quad (8)$$

Then we have the following lemma.

Lemma 8: For each $\alpha = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{r+1}^m$ and $i \in [m]$, the subset $L_\alpha^{(i)} \setminus \{\alpha\}$ is a repair set of α .

Proof: To simplify notation, we assume $i = 1$. Then by assumption of this lemma, we have

$$L_\alpha^{(1)} = \{(\lambda'_1, \lambda_2, \dots, \lambda_m) \in \mathbb{Z}_{r+1}^m; \lambda'_1 \in \mathbb{Z}_{r+1}\}.$$

For each $\lambda'_1 \in \mathbb{Z}_{r+1}$, denote $\alpha_{\lambda'_1} = (\lambda'_1, \lambda_2, \dots, \lambda_m)$. Then $L_\alpha^{(1)} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ and $\alpha = \alpha_{\lambda_1} \in L_\alpha^{(1)}$. Hence,

$$|L_\alpha^{(1)}| = r + 1. \quad (9)$$

For each fixed $\lambda'_1 \in \mathbb{Z}_r = \{0, 1, \dots, r-1\}$, by (5), we have

$$T(\alpha_{\lambda'_1}) = T(\alpha_r) \cup \{1\}.$$

So by (6), we have

$$\begin{aligned} \mathcal{L}(\alpha_{\lambda'_1}) &= \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}_r^m; \mu_1 = \lambda'_1 \text{ and } \mu_j = \lambda_j, \forall j \in T(\alpha_r)\}. \end{aligned} \quad (10)$$

Moreover, by (6), we have

$$\mathcal{L}(\alpha_r) = \{(\mu_1, \dots, \mu_m) \in \mathbb{Z}_r^m; \mu_j = \lambda_j, \forall j \in T(\alpha_r)\}. \quad (11)$$

Combining (10) and (11), we have

$$\mathcal{L}(\alpha_r) = \bigcup_{\lambda'_1=0}^{r-1} \mathcal{L}(\alpha_{\lambda'_1}). \quad (12)$$

By construction of H and \mathcal{C} , for all codeword (x_1, \dots, x_n) of \mathcal{C} , we have

$$x_{\alpha_r} = \sum_{\beta \in \mathcal{L}(\alpha_r)} x_\beta \quad (13)$$

and for each $\lambda'_1 \in \mathbb{Z}_r = \{0, 1, \dots, r-1\}$, we have

$$x_{\alpha_{\lambda'_1}} = \sum_{\beta \in \mathcal{L}(\alpha_{\lambda'_1})} x_\beta. \quad (14)$$

By (10), $\mathcal{L}(\alpha_0), \mathcal{L}(\alpha_1), \dots, \mathcal{L}(\alpha_{r-1})$ are mutually disjoint. So by combining (12), (13) and (14), we have

$$\begin{aligned} x_{\alpha_r} &= \sum_{\beta \in \mathcal{L}(\alpha_r)} x_\beta \\ &= \sum_{\beta \in \bigcup_{\lambda'_1=0}^{r-1} \mathcal{L}(\alpha_{\lambda'_1})} x_\beta \\ &= \sum_{\lambda'_1=0}^{r-1} \left(\sum_{\beta \in \mathcal{L}(\alpha_{\lambda'_1})} x_\beta \right) \\ &= \sum_{\lambda'_1=0}^{r-1} x_{\alpha_{\lambda'_1}} \end{aligned} \quad (15)$$

Note that $L_\alpha^{(1)} = \{\alpha_0, \alpha_1, \dots, \alpha_r\}$ and $\alpha = \alpha_{\lambda_1} \in L_\alpha^{(1)}$. Then by (15), we have

$$x_\alpha = \sum_{\alpha' \in L_\alpha^{(1)} \setminus \{\alpha\}} x_{\alpha'}.$$

Hence, $L_\alpha^{(1)} \setminus \{\alpha\}$ is a repair set of α .

For any $i \in [m]$, by the same discussion, we can prove that $L_\alpha^{(i)} \setminus \{\alpha\}$ is a repair set of α . ■

We give an example as below to show the arguments in the proof of Lemma 8.

Example 9: Let $r = 2$, $m = 6$, $\alpha = (0, 1, 2, 0, 2, 2)$ and $i = 4$. Then we have

$$L^{(i)}(\alpha) = \{\alpha_0, \alpha_1, \alpha_2\},$$

where $\alpha_0 = (0, 1, 2, 0, 2, 2)$, $\alpha_1 = (0, 1, 2, 1, 2, 2)$ and $\alpha_2 = (0, 1, 2, 2, 2, 2)$. By (6), we have

$$T(\alpha_0) = T(\alpha_1) = \{1, 2, 4\} \text{ and } T(\alpha_2) = \{1, 2\}$$

Moreover, by (6), we have

$$\mathcal{L}(\alpha_0) = \{(0, 1, \lambda_3, 0, \lambda_5, \lambda_6); \lambda_3, \lambda_5, \lambda_6 \in \mathbb{Z}_2\},$$

$$\mathcal{L}(\alpha_1) = \{(0, 1, \lambda_3, 1, \lambda_5, \lambda_6); \lambda_3, \lambda_5, \lambda_6 \in \mathbb{Z}_2\}$$

and

$$\mathcal{L}(\alpha_2) = \{(0, 1, \lambda_3, \lambda_4, \lambda_5, \lambda_6); \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in \mathbb{Z}_2\}.$$

So $\mathcal{L}(\alpha_0) \cap \mathcal{L}(\alpha_1) = \emptyset$ and $\mathcal{L}(\alpha_0) \cup \mathcal{L}(\alpha_1) = \mathcal{L}(\alpha_2)$.

Let H be constructed by (7) and \mathcal{C} be the code with parity check matrix H . Then for all $(x_1, \dots, x_n) \in \mathcal{C}$, we have

$$\begin{aligned} x_{\alpha_2} &= \sum_{\beta \in \mathcal{L}(\alpha_2)} x_\beta \\ &= \sum_{\beta \in \mathcal{L}(\alpha_0) \cup \mathcal{L}(\alpha_1)} x_\beta \\ &= \sum_{\beta \in \mathcal{L}(\alpha_0)} x_\beta + \sum_{\beta \in \mathcal{L}(\alpha_1)} x_\beta \\ &= x_{\alpha_0} + x_{\alpha_1}. \end{aligned}$$

So $\{\alpha_0, \alpha_1\}$ is a repair set of α_2 . Similarly, $\{\alpha_1, \alpha_2\}$ is a repair set of α_0 , and $\{\alpha_0, \alpha_2\}$ is a repair set of α_1 .

Now, we can prove Theorem 6.

Proof of Theorem 6: Note that the n coordinates of codewords of \mathcal{C} can be indexed by \mathbb{Z}_{r+1}^m . By Lemma 5, we need to prove that for any $E \subseteq \mathbb{Z}_{r+1}^m$ of size $0 < |E| \leq 2^m - 1$, there exists an $\alpha \in E$ such that α has a repair set $R \subseteq \mathbb{Z}_{r+1}^m \setminus E$. Further, by Lemma 8, it is sufficient to prove that there exists an $i \in [m]$ and an $\alpha \in E$ such that $L_\alpha^{(i)} \setminus \{\alpha\} \subseteq \mathbb{Z}_{r+1}^m \setminus E$. We can prove this claim by induction on m .

Clearly, the claim is true for $m = 1$. To prove the claim for $m \geq 2$, by induction, we can assume that the claim is true for $m - 1$. That is, for any subset $E' \subseteq \mathbb{Z}_{r+1}^{m-1}$ of size $0 < |E'| \leq 2^{m-1} - 1$, there exist an $i \in [m - 1]$ and an $\alpha' = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{m-1}) \in E'$ such that $L_{\alpha'}^{(i)} \setminus \{\alpha'\} \subseteq \mathbb{Z}_{r+1}^{m-1} \setminus E'$, i.e., $(\lambda_1, \dots, \lambda'_i, \dots, \lambda_{m-1}) \notin E'$ for all $\lambda'_i \in \mathbb{Z}_{r+1} \setminus \{\lambda_i\}$. Then we can prove the claim for m as follows.

For each fixed $\lambda \in \mathbb{Z}_{r+1}$, denote

$$E_\lambda = \{(\mu_1, \dots, \mu_{m-1}, \mu_m) \in E; \mu_m = \lambda\}.$$

Clearly, the subsets E_0, E_1, \dots, E_r are mutually disjoint and $\bigcup_{j=0}^r E_j = E$. We have the following two cases:

Case 1: $0 < |E_\lambda| \leq 2^{m-1} - 1$ for some $\lambda \in \mathbb{Z}_{r+1}$. Let

$$E' = (\mu_1, \dots, \mu_{m-1}) \in \mathbb{Z}_{r+1}^{m-1}; (\mu_1, \dots, \mu_{m-1}, \lambda) \in E_\lambda\}.$$

Then $0 < |E'| = |E_\lambda| \leq 2^{m-1} - 1$. By induction assumption, there exist an $i \in [m - 1]$ and an $\alpha' = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{m-1}) \in E'$ such that $(\lambda_1, \dots, \lambda'_i, \dots, \lambda_{m-1}) \notin E'$ for all $\lambda'_i \in \mathbb{Z}_{r+1} \setminus \{\lambda_i\}$. So $(\lambda_1, \dots, \lambda'_i, \dots, \lambda_{m-1}, \lambda) \notin E_\lambda$. Note that E_0, E_1, \dots, E_r are mutually disjoint and $\bigcup_{j=0}^r E_j = E$. Then $(\lambda_1, \dots, \lambda'_i, \dots, \lambda_{m-1}, \lambda) \notin E$ for all $\lambda'_i \in \mathbb{Z}_{r+1} \setminus \{\lambda_i\}$. Let $\alpha = (\lambda_1, \dots, \lambda_i, \dots, \lambda_{m-1}, \lambda)$. Then $\alpha \in E$ and we have $L_\alpha^{(i)} \setminus \{\alpha\} \subseteq \mathbb{Z}_{r+1}^m \setminus E$.

Case 2: $|E_\lambda| \geq 2^{m-1}$ or $|E_\lambda| = 0$ for all $\lambda \in \mathbb{Z}_{r+1}$. Since $0 < |E| \leq 2^m - 1$, there exist a $\lambda_m \in \mathbb{Z}_{r+1}$ such that $|E_{\lambda_m}| \geq 2^{m-1}$ and $|E_\lambda| = 0$ for all $\lambda \in \mathbb{Z}_{r+1} \setminus \{\lambda_m\}$. Hence, $E \subseteq E_{\lambda_m}$ and $E = E_{\lambda_m}$. Now, let $i = m$ and pick an $\alpha = (\lambda_1, \dots, \lambda_{m-1}, \lambda_m) \in E_{\lambda_m}$. Then $(\lambda_1, \dots, \lambda_{m-1}, \lambda'_m) \notin E_{\lambda_m} = E$ for all $\lambda'_m \in \mathbb{Z}_{r+1} \setminus \{\lambda_m\}$. So we have $L_\alpha^{(m)} \setminus \{\alpha\} \subseteq \mathbb{Z}_{r+1}^m \setminus E$.

In both cases, there exists an $i \in [m]$ and an $\alpha \in E$ such that $L_\alpha^{(i)} \setminus \{\alpha\} \subseteq \mathbb{Z}_{r+1}^m \setminus E$.

Thus, by induction, we proved that for any $E \subseteq \mathbb{Z}_{r+1}^m$ of size $0 < |E| \leq 2^m - 1$, there exists an $i \in [m]$ and an $\alpha \in E$ such that $L_\alpha^{(i)} \setminus \{\alpha\} \subseteq \mathbb{Z}_{r+1}^m \setminus E$. By Lemma 8, $R = L_\alpha^{(i)} \setminus \{\alpha\}$ is a repair set of α . Hence, by Lemma 5, \mathcal{C} is an $(n, k, r, t = 2^m - 1)$ -ELRC, which completes the proof. ■

V. CONCLUSIONS

The class of (n, k, r, t) -exact locally repairable codes (ELRC), which permit local repair for up to t erasures by the sequential approach, is the most general setting of LRCs with exact repair. Several subclasses of LRCs that are reported in the literature, such as codes with locality r and availability t , permit local repair for up to t erasures by parallel approach and are contained in the class (n, k, r, t) -ELRC.

The direct product of m copies of the $[r + 1, r]$ single-parity-check code is a family of codes that has locality r and availability m . In this paper, we prove that such codes are in fact an (n, k, r, t) -ELRC with $t = \sum_{i=1}^m i$. We believe that such codes are optimal in term of code rate.

There still remains much work to be done for (n, k, r, t) -ELRC, such as the code rate bound for $t \geq 4$ and the minimum distance bound for $t \geq 3$. Also, constructing (n, k, r, t) -ELRC with sufficiently large code rate (or minimum distance) is an interesting problem.

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